A study of the nonlinear breakage equation: analytical and asymptotic solutions

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Abstract. New solutions of the nonlinear (collisional) breakage equation are given using analytical and asymptotic methods. The dynamic nonlinear breakage equation is transformed to a linear one for some simple forms of the collision kernel; methods for treating the linear equation are employed to obtain solutions for the nonlinear case. Furthermore, it is shown that under particular conditions the particle size distribution can take asymptotically a self-similar form, i.e. the shape of the (appropriately normalized) distribution is independent of time. The self-similar distribution is obtained from the solution of a double nonlinear integral equation. The latter is solved in closed form and numerically (after transformation to a boundary value problem) for simple forms of the collision and breakage kernels; results for the self-similar distribution are presented and discussed.

1. Introduction

A breakage process may be called *nonlinear* if the breakage behaviour of a particle does not depend only on its properties and external forces (as is the case in *linear* breakage) but on the state and properties of the entire system. Nonlinear breakage plays a significant role in many chemical engineering applications, involving relatively dense dispersed systems (e.g. fluidized beds [1]), as well as in various other fields including the size distribution of raindrops [2], the size distribution of asteroids [3], and some types of crushing/milling operations [4]. The simplest case of the nonlinear breakage is the binary collisional one where the fragmentation is effected only through binary collisions between particles. In the atmospheric sciences literature the term *collisional breakage* is used, whereas *spontaneous breakage* is employed for linear breakage [5]. In this paper the more general term nonlinear breakage is preferred in place of collisional breakage to stress the intrinsic features of the problem.

The linear breakage problem has been studied very extensively [6] with regard to its formulation [7], analytical solutions [8], asymptotic approximations [9, 10] and numerical solution [11]; however, this is by no means the case for the nonlinear breakage problem. The latter has been studied in combination with linear breakage and coagulation for raindrop size spectra computation in a very specific case: i.e., when the fragments from all collisional breakage events are just monomers. In the context of asymptotic solution of the linear breakage problem, Cheng and Redner [12] formulated the general nonlinear breakage equation and examined its general behaviour for homogeneous breakage functions. Furthermore, they obtained some asymptotic results for the particle size distribution for the case of equal-size binary breakage. It is worth noting here that the nonlinear breakage equation bears a similarity (due to its quadratic form with respect to particle concentration) to the well known coagulation

equation for which a huge body of literature exists concerning analytical and asymptotic methods of solution [13].

The lack of a systematic treatment of the nonlinear breakage equation has motivated the present study. An attempt in this direction is made here by presenting analytical and asymptotic solutions for simple cases of the nonlinear breakage equation. The results obtained, aside from their inherent value are expected to be useful for interpreting and/or correlating breakage data, or possibly as a benchmark for testing numerical methods required for the solution of realistic (and more complicated) cases.

In section 1 of this paper the general nonlinear breakage equation is formulated in dimensionless form, and comments are made about the properties of the breakage functions. The form of these functions which are used in the rest of the paper is also presented. In section 2 some analytical solutions are derived for simple forms of the breakage functions. In section 3 the asymptotic integral equation for the self-similar (in engineering terminology), or scaling (in physics terminology), particle size distribution function is derived together with the conditions for the existence of self-similarity. The purpose of section 4 is to give analytical or semi-analytical solutions to the above integral equation for some simple breakage functions. Typical examples of the self-similarity distribution are presented and discussed.

2. Problem formulation

The nonlinear breakage process can be described in general by the following nonlinear partial integrodifferential equation:

$$\frac{\partial f'(x', t')}{\partial t'} = \int_0^\infty \int_{x'}^\infty K'(y', z')b'(x', y'; z')f'(y', t')f'(z', t')\,\mathrm{d}y'\,\mathrm{d}z'$$
$$- f'(x', t') \int_0^\infty K'(x', y')f'(y', t')\,\mathrm{d}y' \tag{1}$$

where t' is time, x' is particle volume, $f'(x', t')$ is particle number density distribution, $K'(x', y')$ is the rate of collision between two particles of volume x' and y' respectively, and $b'(x', y'; z')$ is the distribution of particles of volume x' resulting from the breakup of a particle of volume y' due to collision with a particle of volume z'.

Let $f'_0(x') = f'(x', 0)$ be the initial particle size distribution. The total volume concentration, the total number concentration and the mean size of the initial distribution are, respectively:

$$M = \int_0^\infty x f'_0(x)\,\mathrm{d}x \tag{2a}$$

$$N_0 = \int_0^\infty f'_0(x)\,\mathrm{d}x \tag{2b}$$

$$x_0 = \frac{M}{N_0}. \tag{2c}$$

The functions and variables already introduced can be expressed in dimensionless form, as follows:

$$x = \frac{x'}{x_0} \qquad y = \frac{y'}{x_0} \qquad z = \frac{z'}{x_0} \qquad t = K'(x_0, x_0)N_0 t'$$
$$K(x, y) = \frac{K'(x, y)}{K'(x_0, x_0)} \qquad f(x, t) = \frac{x_0 f'(x', t)}{N_0} \qquad b(x, y; z) = x_0 b'(x', y'; z') \tag{3}$$

and equation (1) can be written as

$$\frac{\partial f(x, t)}{\partial t} = \int_0^\infty \int_x^\infty K(y, z) b(x, y; z) f(y, t) f(z, t) \, \mathrm{d}y \, \mathrm{d}z - f(x, t) \int_0^\infty K(x, y) f(y, t) \, \mathrm{d}y. \tag{4}$$

This equation is similar to the linear breakage equation the only difference being that the breakage functions are given as weighted integrals over the entire particle size range, rendering the equation nonlinear.

Remarks about collision kernel and breakage kernel. In general, the collision kernel is the same as the one used in the coagulation theory but without the effectiveness factor. The latter is usually taken as unity in the coagulation theory to facilitate the mathematical study of the coagulation equation [14]. Thus, the above collision kernel is exactly equivalent to that of the coagulation equation and has the same properties. The only restriction on the form of the kernel is that it must be symmetric with respect to its arguments, i.e. $K(x, y) = K(y, x)$. The collision kernel is called homogeneous, with an index of homogeneity λ, if $K(ax, ay) = a^\lambda K(x, y)$.

As regards the breakage kernel, it is an extension of the usual kernel employed in linear breakage theory with an additional parametric dependence on z. This extension reflects the fact that a particle of size y can give quite different fragment distributions upon collision with two different particles of size z_1 and z_2. Generally speaking, the function $b(x, y; z)$ should satisfy the following requirements irrespective of the z value:

(i) Conservation of mass:

$$\int_0^y x b(x, y; z) \, \mathrm{d}x = y. \tag{5}$$

This equation stipulates that the total volume of particles resulting from the breakup of a particle of volume y, must be equal to y.

(ii)

$$\int_0^k x b(x, y; z) \, \mathrm{d}x \geqslant \int_{y-k}^y (y - x) b(x, y; z) \, \mathrm{d}x \tag{6}$$

where $k < \frac{y}{2}$.

This expression states the requirement that only breakage events take place with no rearrangement of mass allowed. It is an absolute condition based on the physical requirement that when breakage occurs such that a particle $x \geqslant y/2$ is formed, the volume contained within the smaller fragments $(y - x)$ must contribute to the total volume of the fragments smaller than $(y - x)$. Further analysis concerning this condition can be found elsewhere [15]. For binary breakage (two fragments per parent particle), the above restriction is simplified being equivalent to a symmetric kernel in the sense $b(x, y; z) = b(y - x, y; z)$.

The above restriction on the form of breakage kernel is very important but it seems to be overlooked in the literature on linear breakage; this leads to physically unrealistic kernels used to fit experimental data. For example, in [16] a kernel is employed that leads to a 0.33 mass fraction of daughter particles in the same class (with ratio of limits 1 : 2) with the parent particle. Obviously this kernel violates the above restriction.

(iii) The number of particles resulting from breakage of a single particle of volume y after its collision with a particle of volume z is given as

$$v(y; z) = \int_0^y b(x, y; z) \, \mathrm{d}x. \tag{7}$$

The breakage kernel can be partially or completely homogeneous. There are two types of partial homogeneity: (*a*) with respect to fragment distribution, i.e. $b(x, y; z) = b(x/y; z)/y$ and (*b*) with respect to the collision event, i.e. $b(x, y; z) = b(x, y; y/z)$. In the case of complete homogeneity the breakage kernel can be written as $b(x, y; z) = b(x/y; y/z)/y$. This case of complete homogeneity can include discontinuous kernels. The simpler discontinuous kernel is one that represents different fragment distributions for the small and the large particle in a collision event; i.e. the form of the breakage kernel is $b_1(x/y)/y$ for $y > z$ and $b_2(x/y)/y$ for $y < z$.

In this paper some simple functions for the collision rate and breakage kernel are employed to obtain analytical and asymptotic solutions to the nonlinear breakage equation. Collision rates of the *product* form $K(x, y) = x^\omega y^\omega$ and the *sum* form $K(x, y) = x^\omega + y^\omega$ are considered. These forms have been used extensively for analytical ($\omega = 1$) [14] and asymptotic [17] solutions of the coagulation equation.

As regards the breakage kernel, a power law form is assumed with the fragment distribution independent of the size of colliding particles. The corresponding kernel has the form $b(x, y; z) = (\nu + 2)(x/y)^\nu/y(0 \geqslant \nu > -2)$ due to restrictions (i) and (ii) [18]. The mean number of fragments per breakage event is $(\nu + 2)/(\nu + 1)$. Also a discontinuous kernel is assumed with random breakage only for the larger of the colliding particles and no breakage at all for the smaller ones, i.e. $b_2(x/y) = 0$.

3. Analytical solutions

As outlined above, the case of a homogeneous breakage kernel independent of z, i.e. of the form $b(x/y)/y$, is treated.

(i) *Constant collision rate, $K(x, y) = 1$.* Substituting into equation (4) and interchanging the order of integration at the right-hand side, leads to

$$\frac{\partial f(x, t)}{\partial t} = -Nf(x, t) + N \int_x^\infty \frac{1}{y} b(x/y) f(y, t) \, \mathrm{d}y \tag{8}$$

where N is the dimensionless total number concentration of particles, defined as $N = \int_0^\infty f(x, t) \, \mathrm{d}x$. The integration of both sides of equation (8) with respect to x, from $x = 0$ to $x = \infty$, gives the following equation for N:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b_0 - 1)N^2 \qquad \text{where} \quad b_0 = \int_0^\infty b(x) \, \mathrm{d}x. \tag{9}$$

The solution of the above equation is $N = (1 - (b_0 - 1)t)^{-1}$. Using the modified time variable

$$\tau = \int_0^t N(t) \, \mathrm{d}t = \frac{1}{1 - b_0} \ln(1 - (b_0 - 1)t) \tag{10}$$

equation (8) is transformed to

$$\frac{\partial f(x, \tau)}{\partial \tau} = -f(x, \tau) + \int_x^\infty \frac{1}{y} b(x/y) f(y, \tau) \, \mathrm{d}y. \tag{11}$$

This is the well known equation of linear breakage with a constant breakage rate. The linearity of the equation, in conjunction with the fact that the variable transformation leaves the initial distribution $f_0(x)$ unchanged, allows the use of the superposition principle. This means that the solution for an arbitrary $f_0(x)$ can be constructed by appropriately superimposing solutions of the equation with a monodisperse initial distribution $\delta(x - 1)$. To obtain

such a solution for a clearly nonlinear problem is gratifying. Thus, the general solution of equation (11) is

$$f(x, \tau) = \sum_{i=0}^{\infty} A^{(i)}(x) \frac{\tau^{i+1} e^{-\tau}}{(i+1)!} + \delta(x-1) e^{-\tau} \tag{12}$$

where

$$A^{(i+1)}(x) = \int_x^1 \frac{l}{y} b(x/y) A^{(i)}(y) \, dy \qquad \text{and} \qquad A^{(0)}(x) = b(x). \tag{13}$$

An important observation is that the total number N diverges at time $t_c = 1/(b_0 - 1)$. The above solution for $f(x, t)$ is not valid for $t > t_c$ because the transformed time τ is not defined. This behaviour implies a shattering transition at $t = t_c$. The system no longer conserves the mass because there is a loss of new phase of infinitesimally small particles. In the case of linear breakage, shattering (if it exists) is independent of time (always $t_c = 0$, [14]). The present behaviour can be easily attributed to the nonlinearity of the process and it is similar to the relevant phenomenon of gelation in the (also nonlinear) coagulation processes where a finite transition (gelation time) exists [19].

The solution of equation (11) for monodisperse initial distribution and power law breakage kernel is [15]

$$f(x, \tau) = e^{-\tau} x^{\nu} \left(\frac{(\nu+2)\tau}{-\ln(x)} \right)^{1/2} I_1(2[\tau(\nu+2)\ln(1/x)]^{1/2}) + \delta(x-1) e^{-\tau} \tag{14}$$

where I_1 is the modified Bessel function of first kind and first order.

(ii) *Product collision rate $K(x, y) = xy$*. Substituting into equation (4), interchanging the order of integration, and using the relation $\int_0^{\infty} x f(x, t) \, dx = 1$, results in the following equation:

$$\frac{\partial f(x, t)}{\partial t} = -x f(x, t) + \int_x^{\infty} b(x/y) f(y, t) \, dy. \tag{15}$$

This is identical to the equation of linear breakage with a linear breakage rate. The superposition principle can be used for the solution of this equation as well. The general solution for a monodisperse initial distribution is obtained from the series (12) but with the real time t in place of the transformed time τ and

$$A^{(i+1)}(x) = \int_x^1 b(x/y) A^{(i)}(y) \, dy + (1-x) A^{(i)}(x). \tag{16}$$

For the product collision rate there is no shattering effect and the above solution is valid for all times. The solution for the power kernel is given in terms of Kummer's hypergeometric confluent function [15]. For the case $\nu = 0$ it takes a very simple form [20]:

$$f(x, t) = (2t + t^2(1-x)) e^{-tx} + \delta(x-1) e^{-t}. \tag{17}$$

4. Self-similarity analysis

The cumulative volume fraction distribution is defined as

$$F(x, t) = \int_0^x x f(x, t) \, dx \tag{18}$$

which, by inversion, leads to

$$f(x, t) = \frac{1}{x} \frac{\partial F(x, t)}{\partial x}. \tag{19}$$

Transforming equation (4) in terms of $F(x, t)$ results in the following equation:

$$\frac{\partial F(x, t)}{\partial t} = \int_x^\infty \int_0^\infty K(y, z) f(z, t) B(x, y; z) \, \mathrm{d}z \, \mathrm{d}F(y, t) \tag{20}$$

where

$$B(x, y; z) = \frac{1}{y} \int_0^x \overline{x} b(\overline{x}, y; z) \, \mathrm{d}\overline{x} \tag{21}$$

is the cumulative volume fraction breakage kernel and represents the fraction of the volume y of a parent particle that belongs to fragments of volume smaller than x after a collision with a particle of volume z.

Assuming that a scaling of the form $c(\xi) = F(x, t)$ holds, where $\xi = x/s(t)$, and substituting in equation (20) with the use of (19) results in

$$-\xi c'(\xi) s'(t) = \int_\xi^\infty \int_0^\infty K(\zeta s(t), z s(t)) B(\xi s(t), \zeta s(t); z s(t)) \frac{c'(z) c'(\zeta)}{z} \, \mathrm{d}z \, \mathrm{d}\zeta \tag{22}$$

where the prime denotes the derivative of a function with respect to its argument.

In what follows it is assumed that the collision rate is homogeneous with a homogeneity index λ, and that the breakage kernel is completely homogeneous. Homogeneity of $b(x, y; z)$ is equivalent to homogeneity of $B(x, y; z)$, i.e. $B(x, y; z) = B(x/y; z/y)$. The homogeneous kernel has been used extensively in linear breakage, not only for a theoretical asymptotic analysis of the problem [9, 10] but also for correlating experimental data from solids grinding [11]. Therefore, it is a natural extension to assume homogeneity for the nonlinear breakage kernel as well. Furthermore, it seems reasonable to consider, as a first approximation, that the outcome of a breakage event depends only on the size ratio of the two colliding particles and not on the absolute value of the particle sizes involved. Thus, equation (22) can be written as

$$-s'(t) s^{-\lambda}(t) = \frac{1}{\xi c'(\xi)} \int_\xi^\infty \int_0^\infty K(\zeta, z) B(\xi/\zeta; z/\zeta) \frac{c'(z) c'(\zeta)}{z} \, \mathrm{d}z \, \mathrm{d}\zeta. \tag{23}$$

Obviously the right-hand side of this equation is time independent, and so must be the left-hand side. Similarly the left-hand side is ξ independent and so must be the right-hand side. This means that equation (23) can be separated by taking each side equal to a constant. The latter may be taken equal to unity without loss of generality. The asymptotic (large-time) solution for $s(t)$ is $s(t) = ((\lambda - 1)t)^{1/(1-\lambda)}$. This solution is meaningful only for $\lambda > 1$, which is a necessary condition for self-similarity to hold. The respective condition for the case of linear breakage is $\lambda > 0$. This difference between the two breakage models has also been noted in [12].

Up to this point, it has been shown that the solution of the homogeneous nonlinear breakage equation can be written as $F(x, t) = c(x t^{-1/(1-\lambda)}/(\lambda - 1))$. The function $c(\xi)$ can be found by setting the right-hand side of equation (23) equal to 1. For simplicity of demonstration, the self-similarity function $\varphi(\xi) = \xi c'(\xi)$ is used in place of $c(\xi)$. Thus, the equation to be solved for φ takes the form

$$\varphi(\xi) = \int_\xi^\infty \int_0^\infty K(\zeta, z) B(\xi/\zeta; z/\zeta) \frac{\varphi(z) \varphi(\zeta)}{z^2 \zeta} \, \mathrm{d}z \, \mathrm{d}\zeta. \tag{24}$$

Having no term independent of the unknown function, equation (24) has, in principle, an infinite number of solutions with most obvious one $\varphi = 0$. To select the required solution, equation (24) must be solved in conjuction with the requirement of total mass conservation which takes the form

$$\int_0^\infty \frac{\varphi(\zeta)}{\zeta} \, \mathrm{d}\zeta = 1. \tag{25}$$

5. Solution of the self-similarity equation

Equation (24) is a homogeneous nonlinear integral equation of the second kind of combined Fredholm and Volterra type. Its solution even with numerical techniques is quite difficult. Nevertheless, some analytical or semianalytical solutions (for specific simple cases) exist and are given here. It is convenient to define the moments of the self-similarity function φ as

$$M_z = \int_0^\infty x^z \varphi(x) \, dx \tag{26}$$

which will be used extensively in what follows.

Case I.

$$K(x, y) = x^\omega y^\omega \qquad B(x/y \cdot z/y) = (x/y)^\mu.$$

The above form of function B corresponds to the previously mentioned power law breakage kernel with $\mu = \nu + 2$. Substituting the above functions into equation (24) and using the variables x, y in place of ξ, ζ for convenience, leads to

$$\frac{\varphi(x)}{x^\mu} = M_{\omega-2} \int_x^\infty y^{\omega-\mu-1} \varphi(y) \, dy. \tag{27}$$

A differentiation of this equation with respect to x results in a linear differential equation which can be readily solved to give

$$\varphi(x) = k x^\mu \exp\left(-M_{\omega-2} \frac{x^\omega}{\omega}\right). \tag{28}$$

k is an integration constant which can be evaluated using the condition (25), so that

$$\varphi(x) = \left(\frac{M_{\omega-2}}{\omega}\right)^{\frac{\mu}{\omega}} \frac{\omega}{\Gamma(\mu/\omega)} x^\mu \exp\left(-M_{\omega-2} \frac{x^\omega}{\omega}\right). \tag{29}$$

However, the moment $M_{\omega-2}$ remains unknown. It can be evaluated by substituting equation (29) in its definition (equation (26) for $z = \omega - 2$), performing the integration and solving the resulting algebraic expression. The final result is

$$M_{\omega-2} = \omega^{(\omega-1)/(2\omega-1)} \left(\frac{\Gamma((\mu-1+\omega)/\omega)}{\Gamma(\mu/\omega)}\right)^{2-1/\omega}. \tag{30}$$

The distribution $\varphi(x)$ for $\mu = 2$ (binary breakage) and several values of ω is shown in figure 1. Obviously, as ω increases the function $\varphi(x)$ tends to shift to larger values of x and to spread out. In figure 2 the function $\varphi(x)$ is depicted for two values of ω and two values of μ ($\mu = 1.5$ for ternary breakage and $\mu = 1.05$ for breakage in 20 fragments). As the number of fragments per breakage event increases the function $\varphi(x)$ tends to shift to smaller x values becoming broader.

Case II.

$$K(x, y) = x^\omega + y^\omega \qquad B(x/y, z/y) = (x/y)^\mu.$$

These functions are substituted into equation (24), which after some algebra yields

$$\frac{\varphi(x)}{x^\mu} = M_{-2} \int_x^\infty y^{\omega-\mu-1} \varphi(y) \, dy + M_{\omega-2} \int_x^\infty y^{-\mu-1} \varphi(y) \, dy. \tag{31}$$
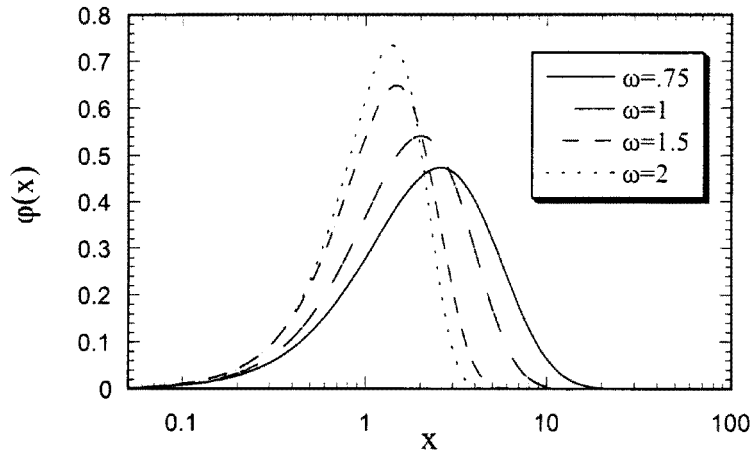
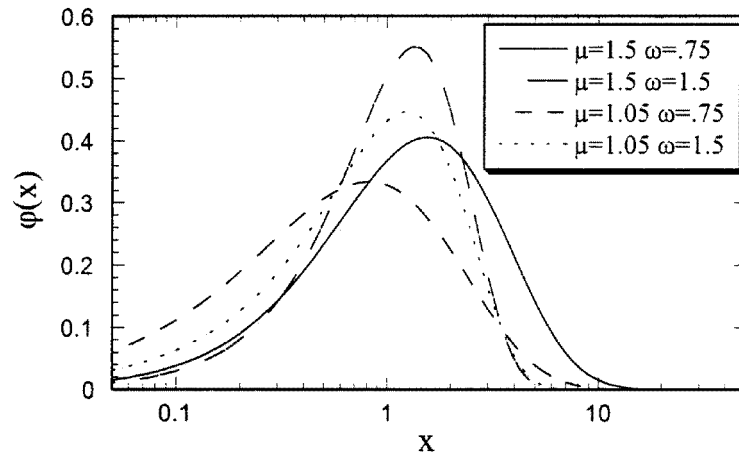**Figure 1.** Self-similarity distribution $\varphi(x)$ (case I) for uniform binary breakage ($\mu = 2$) and several ω values.

Figure 2. Self-similarity distribution $\varphi(x)$ (case I) for several pairs of μ and ω values.

Differentiation with respect to x, and integration of the resulting linear differential equation, for the function $\varphi(x)/x^{\mu}$, leads to

$$\varphi(x) = kx^{\mu - M_{\omega-2}} \exp\left(-M_{-2}\frac{x^{\omega}}{\omega}\right). \tag{32}$$

The integration constant k is again evaluated to satisfy the mass conservation requirement (32). The result is

$$k = \left(\frac{M_{-2}}{\omega}\right)^{\frac{\mu - M_{\omega-2}}{\omega}} \frac{\omega}{\Gamma((\mu - M_{\omega-2})/\omega)}. \tag{33}$$

As in the previous case, the quantities M_{-2} and $M_{\omega-2}$ remain unknown and must be determined from their definition, equation (26). For $z = -2$, one obtains

$$M_{-2} = \left(\frac{\omega\Gamma((\mu - M_{\omega-2} - 1)/\omega)}{\Gamma((\mu - M_{\omega-2})/\omega)}\right)^{\omega/(1+\omega)}. \tag{34}$$
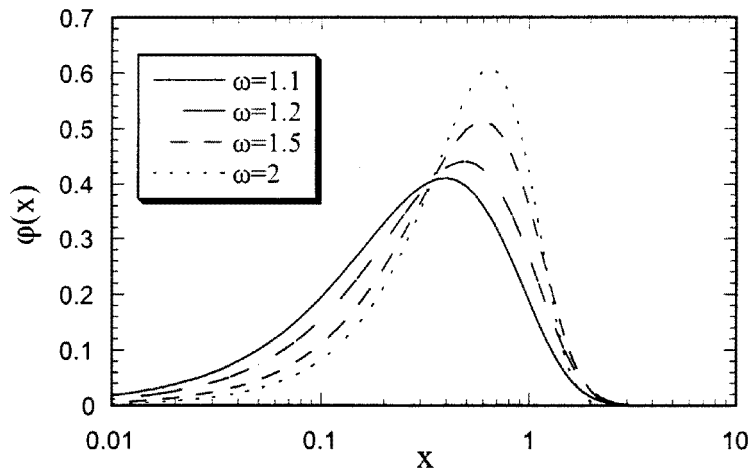
Figure 3. Self-similarity distribution $\varphi(x)$ (case II) for uniform binary breakage ($\mu = 2$) and several ω values.
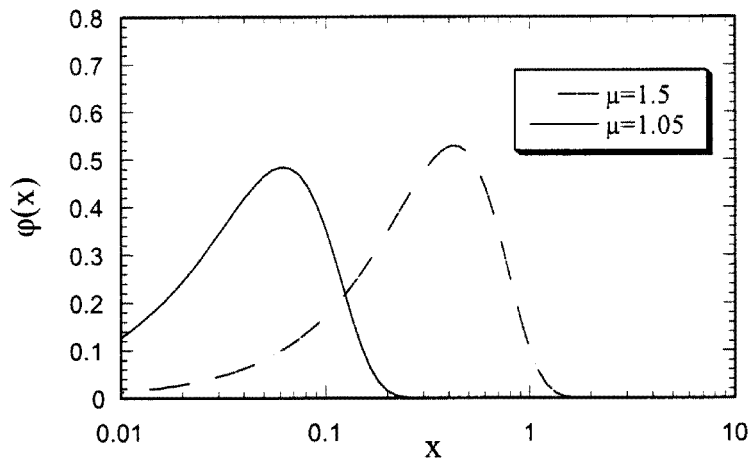

Figure 4. Self-similarity distribution $\varphi(x)$ (case II) for $\omega = 2$ and two μ values.

Similarly, for $z = \omega - 2$, performing the integration and after some algebra, the following trancendental equation is obtained for $M_{\omega-2}$:

$$M_{\omega-2} = \omega^{\frac{\omega-1}{\omega(\omega+1)}} \left(\frac{\Gamma((\mu - M_{\omega-2})/\omega)}{\Gamma((\mu - M_{\omega-2} - 1)/\omega)} \right)^{\frac{\omega-1}{\omega+1}} \left(\frac{\Gamma((\mu - M_{\omega-2} + \omega - 1)/\omega)}{\Gamma((\mu - M_{\omega-2})/\omega)} \right). \tag{35}$$

The function $\varphi(x)$ for $\mu = 2$ and several ω values is shown in figure 3. As ω increases, $\varphi(x)$ tends to larger x value, but unlike case I it becomes narrower. Figure 4 shows $\varphi(x)$ for $\omega = 2$ and two μ values. In this case, as the number of fragments per breakage event increases, $\varphi(x)$ tends to be significantly displaced towards smaller x values, but retains its shape.

Case III. This is the same as case I, with the difference that only the larger particle breaks after a collision event, i.e. $B(x/y, z/y) = 0$ for $z > y$. A rather unclear point must be resolved here. The no-breakage condition for equation (1) corresponds to $b(x, y; z) = \delta(x - y)$. It can

be confirmed by the fact that this rate eliminates the right-hand side of equation (1). However, using directly the definition of $B(x, y; z)$ with respect to $b(x, y; z)$, it is difficult to decide if the no-breakage condition for $B(x, y; z)$ is $B(x, y; z) = 0$ or $B(x, y; z) = \delta(x - y)$. But upon inspection of equation (20) it is clear that the no-breakage condition (elimination of the right-hand side of equation (20)) is indeed $B(x, y; z) = 0$. Substituting in equation (24), and using a new unknown function $A(x) = \varphi(x)/x^{\mu}$, the following equation results:

$$A(x) = \int_x^{\infty} y^{\omega-1} A(y) \int_0^y z^{\omega+\mu-2} A(z) \, dz \, dy. \tag{36}$$

The direct numerical solution of this equation is extremely difficult due to the upper limit of the first integral. To proceed with the discretization, the asymptotic behaviour of $A(x)$ for large x must be known. Fortunately, the above equation can be transformed into a system of ordinary differential equations for which well-established numerical procedures exist. Thus, differentiating equation (36) and using the new variable $D(x) = \int_0^x z^{\omega+\mu-2} A(z) \, dz$ the following system of differential equations results:

$$\frac{dA(x)}{dx} = -x^{\omega-1} A(x) D(x) \tag{37a}$$

$$\frac{dD(x)}{dx} = x^{\omega+\mu-2} A(x). \tag{37b}$$

A third differential equation can be written to facilitate the computation of the total mass of the particle size distribution

$$\frac{dM(x)}{dx} = x^{\mu-1} A(x). \tag{37c}$$

The initial conditions for the solution of the above system of equations (37a)–(37c) is $A(0) = c$ (an arbitrary constant) and $B(0) = M(0) = 0$ (by definition). An additional constraint arises from the requirement for conservation of the total mass as $M(\infty) = 1$. The constant c must be adjusted to satisfy this constraint. This is a typical nonlinear boundary value problem and the best way for its solution is a shooting procedure. The initial value problem is solved with an explicit Runge–Kutta integrator with self-adjusting step and prespecified accuracy [21]. The convergence to the required value of c is achieved using the Newton–Raphson method with numerically computed derivatives. This procedure is extremely effective. The self-similar distribution can be computed with an arbitrary pre-selected accuracy (the only limit being the machine accuracy), taking a few seconds in a modern personal computer. Some typical results, for binary breakage and several ω values, are shown in figure 5. The influence of ω on $\varphi(x)$ is similar to that of case I, but much more pronounced.

Case IV. This is the same as case II but with the difference that only the larger particle breaks after a collision event, i.e. $B(x/y, z/y) = 0$ for $z > y$. Substituting into equation (24) and using the new unknown function $A(x) = \varphi(x)/x^{\mu}$ one obtains

$$A(x) = \int_x^{\infty} y^{\omega-1} A(y) \int_0^y z^{\mu-2} A(z) \, dz \, dy + \int_x^{\infty} y^{-1} A(y) \int_0^y z^{\mu+\omega-2} A(z) \, dz. \tag{38}$$

Differentiating with respect to x and introducing the new variables $D(x) = \int_0^x z^{\mu-2} A(z) \, dz$ and $E(x) = \int_0^x z^{\mu+\omega-2} A(z) \, dz$ leads to the system of ordinary differential equations

$$\frac{dA(x)}{dx} = -x^{\omega-1} A(x) D(x) - x^{-1} A(x) E(x) \tag{39a}$$

$$\frac{dD(x)}{dx} = x^{\mu-2} A(x) \tag{39b}$$

$$\frac{dE(x)}{dx} = x^{\mu+\omega-2} A(x). \tag{39c}$$
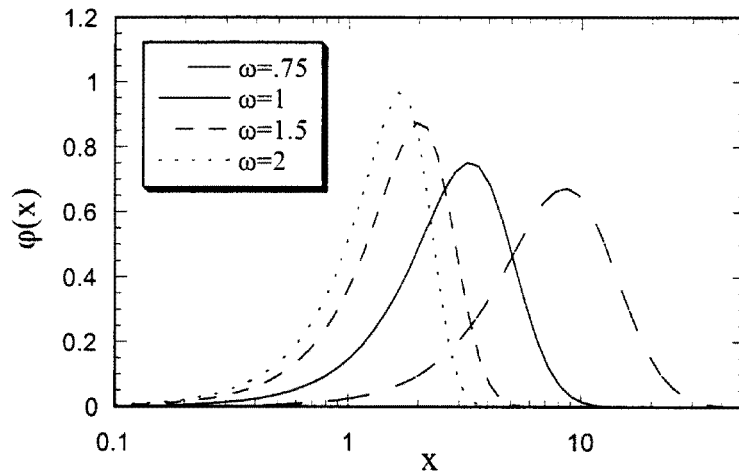
Figure 5. Self-similarity distribution $\varphi(x)$ (case III) for uniform binary breakage ($\mu = 2$) and several ω values.
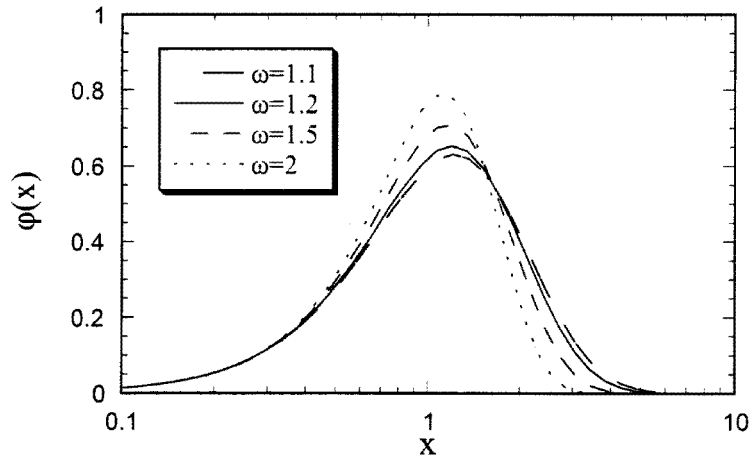

Figure 6. Self-similarity distribution $\varphi(x)$ (case IV) for uniform binary breakage ($\mu = 2$) and several ω values.

Equation (37c) must be added for the computation of the total mass. An asymptotic treatment of this system in the region of $x = 0$ shows that for $\mu + \omega > 2$ (which is always valid for breakage in a finite number of fragments) $(\frac{dA}{dx})_{x=0} = 0$. This is a major difference compared to case II (splitting of both colliding particles) where the function $A(x)$ has a singularity at $x = 0$ (see equation (32)). The above condition allows one to use the initial condition $A(0) = c$ where c is an arbitrary constant. The other initial conditions are $M(0) = D(0) = E(0) = 0$ (from the definition of the respective functions). The constant c must be adjusted to satisfy the condition $M(\infty) = 1$. The numerical treatment here is similar to that of case III, except that there is a system of four equations here instead of three. The function $\varphi(x)$ for binary breakage and several values of ω is plotted in figure 6, showing that the influence of ω on $\varphi(x)$ is rather small. The function $\varphi(x)$ is shifted to larger x values compared with case II (breakage of both colliding particles).

6. Concluding remarks

In this paper the general nonlinear (collisional) breakage equation has been studied. The restrictions on the form of the breakage functions are discussed and some analytical solutions are obtained for simple cases of these functions. From the solutions it appears that the shattering behaviour of nonlinear breakage is closer to the kinetics of gelation of the coagulation equation than to the kinetics of shattering of the linear breakage. Obviously, this is a consequence of having the same order of nonlinearity as the coagulation equation.

It is also shown that the nonlinear breakage equation may admit self-similar solutions for homogeneous breakage functions. Analytical forms of the self-similar solutions are offered for some types of homogeneous breakage kernels. The functional form of this self-similarity distribution is reminiscent of that of the linear breakage similarity theory [10] (modified Gamma distribution). For discontinuous breakage functions the problem is transformed to a nonlinear boundary value problem which has a straightforward numerical solution. Some typical results for the self-similar distributions are given and discussed.

The solutions of the nonlinear breakage equation obtained in this paper can serve the development of general numerical procedures which, in combination with realistic breakage kernels, are necessary for the assessment and interpretation of experimental data in various fields.

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